Grading guide, Pricing Financial Assets, June 2018

1. Consider a commodity with current price S_0. Assume that it can be stored, that there are commodities at storage (so that it may be sold), and that the net storage costs of the commodity can be described as a constant proportional continuously paid cost of δ.
   (a) Describe the workings of a forward contract on the commodity.
   (b) Assume a constant continuously compounded risk-free interest rate of r. Using an arbitrage argument find the forward price F_0 at time 0 on the forward contract that matures at time T *Hint: You may think at an analogy with a stock paying a continuously paid dividend rate.*
   (c) Consider the value of a forward contract with forward price K at time $t < T$ and assume that the commodity price follows the geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dz$$

   Using Ito's lemma find the process followed by the value of the forward contract. What will the drift be under the risk neutral measure?

Solution:

(a)

Definition 0.1 (Forward Contract). *A* ***Forward Contract*** *is an agreement to buy or sell a quantity of an asset of a specified quality and type (called* ***the underlying asset****) delivered at a specified future time T (****the expiration date****) and at a specified place for an agreed* ***delivery price*** *K as measured in some defined numeraire/currency.*

The price is initially (at t) set at a level $K = F(t, T)$, such that the value $V_K(t, T)$ of the contract is 0, i.e. by definition

$$V_{F(t,T)}(t, T) = 0$$

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The price $F(t, T)$ is called ***the forward price*** *at t for delivery at T.*

(b) Cf. e.g. Hull (8ed) p. 312-3 and p.292. Since the commodity can be stored there is an arbitrage between spot and forward purchase, and the argument is analogous to the case of a stock with a continuously paid dividend rate: Buy the commodity spot financed by a loan at an interest of r and accumulate interest and storage costs. The accumulated liability at T will be $S_0 \mathrm{e}^{(r+\delta)T}$, and you have the commodity. An alternative is to enter into a forward contract with delivery at T. To eliminate arbitrage it must be that the forward price at 0 for delivery at T is $F_0 = S_0 \mathrm{e}^{(r+\delta)T}$

(c) For a forward contract with forward price K the value is $V_K = S_t \mathrm{e}^{\delta(T-t)} - K\mathrm{e}^{-r(T-t)}$. Using Ito's lemma you get that the differential form for the value process for the forward:

$$dV_K = [(\mu - \delta)\mathrm{e}^{\delta(T-t)} S_t - rK\mathrm{e}^{-r(T-t)}]dt + \sigma S_t \mathrm{e}^{\delta(T-t)} dz$$

For a currently priced forward (i.e. where K gives the forward contract a zero net present value $K = F_t = \mathrm{e}^{(r+\delta)(T-t)} S_t$) we see that the drift is 0 under the risk neutral measure (by setting $(\mu - \delta) = r$), while a non-zero market value will grow at a rate r. It will be considered ok just to derive this for the currently priced forward.

2. Suppose certain derivatives have values that depend on a single state variable given by the process

$$\frac{d\theta}{\theta} = mdt + sdz$$

where dz is a Wiener process.

(a) Consider two such derivatives, assume a continuously compounded risk-free interest rate of r, and use an arbitrage argument to derive and define the market price of ($\theta-$)risk, λ (You may assume that the prices of the derivatives follow geometric Brownian motions).

(b) If θ is itself a traded asset, what can we say about the relation between m, s and the market price of risk?

Solution:

(a) Consider two derivatives with prices following two different GBM's with the same underlying Wiener process dz. Use these to form a locally risk free portfolio, and using that this, barring arbitrage, must return the risk free rate you can derive the market price per unit of risk (λ) as the drift of the derivative price in excess of the risk free rate divided by the volatility, i.e. of the form

$$\frac{\mu - r}{\sigma} = \lambda$$

. This is the same for both, hence all such, derivatives (Hull, 8ed, section 27.1).

(b) If θ is itself a traded asset, then (in arbitrage equilibrium)

$$\frac{m - r}{s} = \lambda$$

too.

3. Suppose a company i is in default before or at time t with probability $Q_i(t)$.

(a) We will often assume that $Q_i(t)$ is weakly increasing in t. Why?

(b) In a model of defaults a random variable x_i is used to indicate a default before or at time T. It is assumed that x_i has a standard normal distribution, and that a default occurs at or before time T if

$$\Phi(x_i) \leq Q_i(T)$$

where Φ denotes the standard normal cumulative distribution function. Interpret this. Why is such a model of computational value?

(c) In the Vasicek one-factor model of credit risk the default indicators x_i for different companies $i = 1, \ldots, n$ are given by

$$x_i = a_i F + \sqrt{1 - a_i^2} Z_i$$

where the a_is are constants, and F,$(Z_i)_{i=1,\ldots,n}$ are independent standard normal random variables. Describe and interpret this model. When can it be a reasonable description of default?

Solution:

(a) This follows directly from the definition. The assumption in the model is that if a company defaults it stays in default.

(b) It is easier to run simulations using the random variable that is normally distributed, and especially easier to simulate correlated defaults when the marginal distributions are standard normal, rather that complicated empirical distributions.

(c) Cf. Hull p.538-40 Note that F is a common factor determining the covariance structure, whereas Z_i are ideosyncratic factors, and a_i are sensitivities to the common factor. Note that $Var[x_i] = a_i^2 + (1 - a_i^2) = 1$ and $Cov[x_i, x_j] = a_i a_j$ using that F and Z_i are standard normal and independent. In implementation the a_i are often approximated as the correlation between i's equity return and some broad market index (i.e. $\mathbb{P}$-type correlations). They could also be inferred from e.g. prices of tranched CDOs (i.e. $\mathbb{Q}$-type correlations)).